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Non-perturbative string equations for type 0A

Clifford V. Johnson*

Department of Physics and Astronomy, University of Southern California†
Los Angeles, CA 90084-0484, U.S.A., and
Kavli Institute for Theoretical Physics, University of California
Santa Barbara, CA 93106-4030, U.S.A.
E-mail: johnson1@usc.edu, johnson1@kitp.ucsb.edu

Abstract: Well-defined non-perturbative formulations of the physics of string theories, sometimes with D-branes present, were identified over a decade ago, from careful study of double scaled matrix models. Following recent work which recasts some of those early results in the context of type 0 string theory, a study is made of a much larger family of models, which are proposed as type 0A models of the entire superconformal minimal series coupled to supergravity. This gives many further examples of important physical phenomena, including non-perturbative descriptions of geometrical transitions between D-branes and fluxes, tachyon condensation, and holography. In particular, features of a large family of non-perturbatively stable string equations are studied, and results are extracted which pertain to type 0A string theory, with D-branes and fluxes, in this large class of backgrounds. For the entire construction to work, large parts of the spectrum of the supergravitationally dressed superconformal minimal models and that of the gravitationally dressed bosonic conformal minimal models must coincide, and it is shown how this happens. The example of the super-dressed tricritical Ising model is studied in some detail.

Keywords: Bosonic Strings, D-branes, String Duality, Conformal Field Models in String Theory.

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*On leave from the Centre for Particle Theory, Department of Mathematical Sciences, University of Durham, Durham DH1 3LE, England.

†Address for correspondence.

Contents

1. Opening Remarks

Between the First and Second Superstring Revolutions, there was another burst of excitement, based on the technology of large N matrix techniques. This "Matrix Revolution" has generally been regarded as having delivered much less than it seemed to promise.[1]. Occasionally it is still referred to as having taught us a few of the early lessons about important features of string theory beyond perturbation theory (see e.g., ref. [1]), but further than that not much else has been said. It has been suggested by some over recent years that there are more useful lessons to be learned from that period, some of which may be relevant to current issues (see e.g., ref. [2]). Admittedly, most of the suggestions had not been bolstered by computational evidence in a definite framework of ideas, and so were largely ignored. This year has seen a number of key papers [3]–[7] which have supplied both framework and computations renewing interest in connecting the old results to new ideas, and thereby refurbishing an old laboratory for the testing of more modern ideas. In short, one-dimensional $N\times N$ matrix models have been identified as the world-volume theory of N of the D-branes the appropriate continuum theory [8] (Liouville coupled to a scalar), thereby recasting the large N physics as another tractable example of holography since the string theory is two-dimensional. Furthermore, since the matrix model potential is recognised as the open string world-volume tachyon potential, we have a powerful new setting within which to study tachyon condensation [9]–[12], the understanding of which is

[1]Not unlike the recently released film with a similar title.

of considerable importance. For these and other reasons (see below) it is very clear that the double scaled technology has delivered on many of its promises and may well continue to do so as we explore it with fresh eyes.[2]

One of the biggest promises on which the double-scaled matrix model technology is alleged to not have delivered was that of supplying a complete and well-defined non-perturbative definition of the stringy physics. This assertion has in fact been known to be untrue, but by only a few (despite having been present in the literature for a dozen years). It is extremely simple and natural[3] to define string theories with exactly the same genus by genus expansion as that provided by the original matrix models [21]–[24], but with a perfectly well defined and unique non-perturbative completion [13]–[15].

It is also often remarked that it is only recently that we have achieved understanding of Dirichlet open string sectors (i.e., D-branes) in the context of those early string models. Actually, perfectly sensible perturbative definitions were derived at an early stage [25, 26], and the non-perturbative definition of these closed string theories mentioned in the previous paragraph was extended in ref. [27] to achieve the inclusion of world sheets with Dirichlet boundaries. This was used to show [27] that the double-scaled unitary matrix models [28, 29] were not supplying the physics of some mysterious continuum theory (another misconception with surprising longevity), but were just a rewriting of string theories, albeit with the inclusion of open string sectors.

It is worth noting that despite the excitement and remarkable successes of the Second Revolution, which gave us so much insight into strings beyond perturbation theory, the bulk of the non-perturbative information which we have is still phrased very much in terms of perturbative string theory or field theory, by invoking strong-weak coupling dualities. It is still difficult to supply definitions of physics at intermediate values of the coupling, without restricting attention to elementary considerations such as the spectra of BPS states. The $c \leq 1$ physics supplied by the double-scaled matrix model techniques years ago was in terms of families of non-linear differential equations called "string equations" in the old literature [21]–[24]. The solutions of these equations obtained by asymptotic expansion correspond to weak string coupling perturbation theory, but they encode physics at arbitrary values of the coupling in the full solutions. These systems may well contain useful lessons about string theory at intermediate coupling which may help address issues of current concern.

The reason for recalling the various facts listed above is so that we may exhibit them as components of a possible series of further valuable lessons which the Matrix Revolution taught us which were missed the first time around. Perhaps it was necessary for the field to

[2]Sadly, the same cannot be said for the film mentioned in the previous footnote.

[3]The definition [13]–[15] naturally arose first in studies of double-scaled complex matrix models, but were quickly shown to be realised in any double scaled matrix model, such as unitary or hermitean, where parameters can be tuned to permit the merger of two disconnected components of the Dyson gas of eigenvalues. The original complex matrix model scenario, which contains a natural "wall" into which a single component can be tuned to collide, is just a $\mathbb{Z}_2$ identification of a picture with two symmetric components. Matrix models which appeared later (in the $c = 1$ matrix model context) with such doubled distributions [16]–[18] of eigenvalues, have recently been reinterpreted as type 0 string theories [19, 20].

depart and return to the results from another direction, enriched with the lessons learned elsewhere, such as the importance of branes in getting access to strongly coupled physics, the significance of large N in other stringy contexts such as holography, etc. Quite a few of these features are evident in the double scaled matrix models, but of course they are hard to see because there is not much room for physics to happen in these low dimensional scenarios.

A recent paper [30] has gathered considerable evidence for a picture in which many of the aforementioned facts fit together in a way which is highly instructive, and quite exciting. First, the string theories for which the well-defined non-perturbative formulation was provided are not the simple bosonic string theories but in fact the type 0 string theories (although there are key similarities; see later). Specifically, the type 0B string theories of the $(2, 2m)$ superconformal minimal models coupled to super-Liouville theory are captured by unitary one-matrix models (or equivalently two-cut hermitean one-matrix models), while the type 0A theories of the same worldsheet models are captured by the complex one-matrix models (or symmetric two-cut hermitean one-matrix models, or equivalently the sector of even perturbations of the unitary one-matrix models). The physics of these models is captured by string equations which supply unique and sensible non-perturbative data. In this new context, some of the new lessons we might learn from matrix models immediately begin to become apparent. For example, the interpretation of the inclusion of the open string sectors already supplied in the earlier work [27] becomes extended to include Ramond-Ramond flux as well, and the non-perturbative physics supplied by the string equations connects the two, since they arise as different asymptotic expansions of the same equation. In fact, such an elegant non-perturbative encapsulation of geometric transitions [31]–[35] between open string (D-branes) and closed string (fluxes) physics is highly sought after in the modern context, and for at least that reason it is worth revisiting the old models, now dressed in their new clothes.

1.1 The purpose of this paper

The purpose of the work reported on in this paper is to begin to define and explore more models in this new context, and study new and more complicated examples. As stated above, the work of refs. [13]–[15] in defining non-perturbative string physics for all the $(2, 2k - 1)$ conformal models coupled to Liouville theory, extended further in ref. [27] to naturally include open string sectors in the non-perturbative discussion while also interpreting doubled unitary scaled matrix models in these terms, was recently recast in terms of type 0A string theories (with maps to type 0B in (at least) one case, using the result of ref. [36, 37]), where the relevant conformal models are the $(2, 2m)$ superconformal series, where $m = 2k$. In particular, for type 0A, the string equation is that of refs. [37] and [13]–[15]:

$$u\mathcal{R}^2 + \frac{1}{2}\mathcal{R}\mathcal{R}'' - \frac{1}{4}(\mathcal{R}')^2 = \nu^2\Gamma^2\,, \qquad \text{where} \qquad f' \equiv \nu\frac{\partial f}{\partial z}\,, \tag{1.1}$$

and ν is a parameter related to the closed string coupling, since it is the renormalised $1/N$ which survives the double scaling limit of the $(N\times N)$ matrix model. Furthermore,

$$\mathcal{R}=\sum_{l=1}^{k}\left(1+\frac{l}{2}\right)t_l R_l[u]-z\,, \tag{1.2}$$

where the t_l, $l=1,\ldots,k$, are couplings to the gravitationally dressed conformal field theory operators, $\mathcal{O}_l$, in the theory and the $R_l[u]$ are polynomials in u and its derivatives with respect to z, given below in equation (2.7). The function u is the two point function of the operator $\mathcal{O}_0$, which couples to $z=-t_0$, and so the free energy can be obtained by integrating u, as the precise relation is:

$$u=-2\nu^2\frac{\partial^2 F}{\partial z^2}\,. \tag{1.3}$$

In ref. [27], it was shown that the large positive z expansion of a solution to equation (1.1) has an interpretation in terms of string world-sheet perturbation theory in the presence of Γ branes, since Γ appears raised to the power b, where b is the number of boundaries in the genus expansion. For $\Gamma=0$, the perturbation expansion is identical to that of the string equations of refs. [21]–[24] which have the ill-defined non-perturbative physics for k even. Indeed, those equations are simply contained in the above equation as the case $\mathcal{R}=0$. For example, in the case of pure gravity, the $k=2$ model, we have:

$$u(z)=z^{1/2}+\frac{1}{2}\frac{\nu\Gamma}{z^{\frac{3}{4}}}-\frac{1}{24}\frac{\nu^2}{z^2}(6\Gamma^2+1)+\frac{15}{256}\frac{\nu^3}{z^{\frac{13}{4}}}\Gamma(4\Gamma^2+3)-\frac{7}{4608}\frac{\nu^4}{z^{\frac{9}{2}}}\left(180\Gamma^4+345\Gamma^2+28\right)+\cdots\,. \tag{1.4}$$

On integrating to get the free energy, it is seen that the natural dimensionless topological expansion parameter (closed string coupling) is $\kappa=\nu/z^{5/4}$. Worldsheets with h handles and b boundaries come with a factor $\kappa^{2h-2+b}\Gamma^b$, and so Γ naturally has an interpretation as the number (in some units[4]) of D-branes to which each boundary couples.[5]

That the equation (1.1), for $\Gamma=0$, has a sensible large negative z expansion, in contrast to the case $\mathcal{R}=0$, was key to the realisation of refs. [13]–[15] that there is sensible non-perturbative formulation. In fact, for arbitrary Γ, the $z\to-\infty$ expansion has a sensible world-sheet interpretation, and the regions of intermediate z connect smoothly through $z=0$ to each other in a unique[6] way, supplying a complete non-perturbative definition of

[4]In fact, for the purposes of this paper, the precise normalisation of Γ does not matter too much. It is enough to know that Γ simply counts the number of D-branes, or half R-R flux insertions (see later), in some units. All that is important here is its exponent in various equations, which help organise perturbation theory. This was why Γ was called the "open string coupling" in ref. [27].

[5]The observation in ref. [27] that double scaled unitary models of ref. [28, 29] are just the usual string theories with open string sectors was based on realising that solutions of the string equations obtained in that context can be mapped to solutions of equation (1.1), with the boundary conditions such that their expansion is of the form (1.4), for a specific value of Γ.

[6]Uniqueness is suggested by numerical work [13]–[15], and by direct proof in special cases [38]. Furthermore, setting $\Gamma=1/2$, this equation can be mapped using the Miura map to the family of solutions to the string equations of the double-scaled unitary matrix models with even potentials. These are known to be unique from analytic studies. See also the previous footnote.

the physics. For example, for $k=2$ again, we have:

$$u(z) = \frac{1}{4}\frac{\nu^2}{z^2}(4\Gamma^2-1) + \frac{1}{32}\frac{\nu^6}{z^7}(4\Gamma^2-1)(4\Gamma^2-9)(4\Gamma^2-25)\times \times\left(1+\frac{11}{96}\frac{\nu^4}{z^5}(48\Gamma^4-1240\Gamma^2+8371)\right)+\cdots. \quad (1.5)$$

In the interpretation in terms of open strings, it is strange that there are no world sheets with an odd number of boundaries, and no interpretation was given in the earlier work for the missing orders, although a connection with the "topological" solution $u=\nu^2/(4z^2)$ was suggested, since for all k this is the leading term in all the large negative z expansions. In the new setting of ref. [30], this large negative z expansion has a new interpretation: The string theory is closed and Γ represents an even number of insertions due to the presence of Γ units of R-R flux. Since the two expansions — one with D-branes and one with R-R flux — are connected smoothly through the strong coupling region by the full string equation (1.1), this gives a complete and elegant example of geometric transitions between descriptions with D-branes and those with R-R fluxes.

There is a large family of natural generalisations of the string equation (1.1). These were presented in refs. [39], where the context was the full family of (q,p) conformal minimal models coupled to Liouville theory. The principal techniques used there were the underlying structure of the generalised KdV hierarchy which organises the physics. The inclusion of open string sectors was derived in ref. [40], by exploiting further the underlying integrable system, and appealing to the elegant organisation of the key structures in terms of integrable systems and the modes of a twisted boson. Lest fears arise that a loss of contact with a matrix model and hence a worldsheet description may have resulted from paying too much attention to the integrable system to derive these generalisations, it should be noted that the open-closed string equation of ref. [40] for the case $(4,3)$ was reproduced in an explicit two-matrix model computation in ref. [41].

In this paper these more general equations, presented in section 2, will be explored somewhat further than they were when originally presented, in the light of the new setting. We will make a number of observations about the physics they contain. Part of the goal is to make the natural suggestion that they are the equations defining the type 0A system of the full (q,p) superconformal minimal models coupled to super-Liouville theory. For this to work, it is evident that there must be a striking coincidence between the spectrum of dressed operators in the superconformal theory and those in the bosonic theory, for which the string equations were originally presented. This coincidence was already made explicit for the $(2,2k-1)$ (bosonic) *vs.* the $(2,4k)$ (supersymmetric) case presented in ref. [30], and it is natural to conjecture that it is more generally true. In section 3 we show how it works in general.

In section 4, we study a specific example, which is in fact to be interpreted as the tricritical Ising model not with the ordinary gravitational dressing, but *super*gravitationally dressed.[7] We observe some features of the perturbative expansions and anticipate that the full non-perturbative solution is well defined. Our observations strongly suggest that there

[7]In fact, it now wears a cape.

JHEP03(2004)041

is a rich family of geometrical transitions between D-branes and R-R fluxes, all of which the model has a complete non-perturbative description. There also seem to be possible solutions with a novel interpretation in terms of R-R flux in *both* weak coupling regimes for which D-branes can be switched on in one weak coupling region by switching on a further operator. This operator induces a flow to an open string generalisation of what is the ordinary Ising model in the bosonic context. We note that more work is needed to study the non-perturbative solutions of the string equations, since our computations have been centered around exploring the possible weak coupling limits of the string equation, and anticipating that at least a subset of the behaviours that we find are connected to each other by the non-perturbative physics contained in the equation. Such further study to explore the fate of the solutions we have found is likely to be highly instructive and fruitful.

Finally, the non-trivial numbers appearing in the one-loop partition function of the $q = 3$ series of models are noted, since they are clues to confirming the identity of the models. Unfortunately, we cannot conclude anything about them until the corresponding numbers for the type 0B model are known, since the contributions from the two sets of spin structures must be added to compare with known results. Formulating the (q, p) 0B models is beyond the scope of this paper, and so a full comparison is left to the future. There is an interesting feature however, extending what was observed for the $q = 2$ case long ago [27], and interpreted in the present context in ref. [30]: A large class of solutions with flux interpretation in the large z expansion all have vanishing contribution on the sphere, and the value of the torus contribution is universal to all models in the 3-series. This is likely to be true in the general (q, p) string models.

2. More general string equations

The purely closed string (q, p) generalisations of the $(2, 2k - 1)$ string equation (1.1) were presented in ref. [39], although the fullest discussion of properties and structure of the entire family of equations and their associated matrix models can be found in ref. [42]. A summary follows, and then the open string generalisation, derived in ref. [40] will be stated.

2.1 Closed strings

To appreciate the structure of the equations, it is most useful to have in mind the structure of the generalised KdV hierarchies. The reader not interested in this structure can simply skip to the bottom of this subsection.

The qth KdV hierarchy can be formulated in terms of a qth order differential operator Q, of the form:

$$Q = d^q + \sum_{i=2}^{q} \alpha_i \{u_i, d^{q-i}\}\,, \tag{2.1}$$

where there are $q - 2$ functions u_i, and d here denotes (ν times) the derivative with respect to z. The α_is are normalisation constants. The hierarchy is a family of flow equations:

$$\frac{\partial Q}{\partial t_r} = \left[Q_+^{r/q}, Q\right], \tag{2.2}$$

where $Q_+^{1/q}$ is constructed using the elegant technology of pseudo-differential operators [43]. Briefly, $Q^{\frac{1}{q}}$ is an infinite series:

$$Q^{1/q} = d + \sum_{j=1}^{\infty} S_j[u_i]d^{-1-j}\,, \tag{2.3}$$

where the operation d^{-1} is defined by:

$$d^{-1}u = \sum_{i=0}^{\infty}(-1)^i u^{(i)} d^{-1-i}\,, \tag{2.4}$$

where $u^{(i)}$ means $u(z)$ differentiated i times with respect to z. The functions S_j in the above are polynomials in the u_i and their z-derivatives. With the above definition, is easy to raise $Q^{\frac{1}{q}}$ to the desired integer power r, and then $Q_+^{\frac{r}{q}}$ denotes the truncation of the result by discarding terms containing odd powers of d.

Of course, if $r = 0 \bmod q$, the flows defined in equation (2.1) are trivial, and so we only consider the case when q does not divide r. Hence, we write $r = qk + l$, where $l = 1, \ldots, q-1$, and $k = 0, \ldots, \infty$. The generalised times t_r will be written $t_{l,k}$.

Just for orientation, the more familiar KdV hierarchy is the case $q = 2$, for which there is just one index to vary, k, which gives the kth flow for the function $u_2 = -u$:

$$\frac{\partial u}{\partial t_k} = \mathcal{R}'_{k+1}\,, \tag{2.5}$$

where the $\mathcal{R}_k$ are given in equation (2.7). So in the more general case of $q > 2$, one can think of l as parameterising $q-1$ different towers of flows, where k denotes where one is on the tower.

After some work, one can write the general flow equations directly as differential equations for the u_i:

$$\alpha_i \frac{\partial u_i}{\partial t_{l,k}} = \mathcal{D}_1^{ij} R^j_{l,k+1} = \mathcal{D}_2^{ij} R^j_{l,k}\,, \qquad i,j = 2,3,\ldots q\,, \tag{2.6}$$

where there's no sum on i intended, and the α_i are constants. The derivative operators $\mathcal{D}_{1,2}$ arise from the "bi-hamiltonian" Poisson bracket structure of the the KdV systems, and define recursion relations between the differential polynomials $R^i_{l,k}$ *via* the second relation in equation (2.6). These polynomials in the $\{u_i\}$ and their derivatives are generalisations of the famous Gel'fand-Dikii[44] polynomials which appear in the usual KdV system, where $Q = d^2 - u$, and:

$$\begin{aligned} R_0 &= 2\,, \qquad R_1 = -u\,, \qquad R_2 = \frac{1}{4}\left(3u^2 - u''\right), \\ R_3 &= -\frac{1}{16}\left(10u^3 - 10uu'' - 5(u')^2 + u^{(4)}\right), \ldots, \end{aligned} \tag{2.7}$$

and the derivative operators in this $q = 2$ case are:

$$\mathcal{D}_1 = d\,, \qquad \mathcal{D}_2 = \frac{1}{4}d^3 - \frac{1}{2}u' - ud\,, \tag{2.8}$$

so that $\mathcal{D}_1 R_{k+1} = \mathcal{D}_2 R_k$. In general, the $R^i_{l,k}$ are fully determined by acting with the $\mathcal{D}_{1,2}$ operators, once the seed constants are fixed. In our normalisation, $R^j_{1,0} = q\delta_l^{j-1}$. An explicit example of the case $q = 3$ showing how the more general structure works, will be given shortly.

With all of the structures in place, it is quite simple to state the general form of the string equations. The following set of $q-1$ equations:

$$\mathcal{D}_2^{ij}\mathcal{R}^j = 0\,, \tag{2.9}$$

where

$$\mathcal{R}^i = \sum_{l=1}^{q-1}\sum_{k=0}^{\infty}\left(k+\frac{l}{q}\right)t_{l,k}R^i_{l,k} = \sum_{l=1}^{q-1}\sum_{k=1}^{\infty}\left(k+\frac{l}{q}\right)t_{l,k}R^i_{l,k} + (i-1)t_{i-1,0}\,, \tag{2.10}$$

may be combined into a single equation by multiplying on the left by $\mathcal{R}^i$ and summing. The result is in fact a total derivative. The equation resulting from integrating once and setting the constant to zero is the string equation:[8]

$$\int dz\, \mathcal{R}^i\mathcal{D}_2^{ij}\mathcal{R}^j = 0\,. \tag{2.11}$$

This elegant compact form may not be totally illuminating to the reader, so it is worthwhile to see how the $q = 2$ string equation (1.1) arises. It can be done by eye, looking at $\mathcal{D}_2$ defined in (2.8), with $\mathcal{R}$ given in (1.2), one gets a total derivative, $\mathcal{R}\mathcal{D}_2\mathcal{R}$, which can be integrated once to give equation (1.1). To match onto closed string perturbation theory, which is equivalent to $\mathcal{R} = 0$, the constant must be set to zero.

2.2 Open strings

As might be guessed from an examination of the $q = 2$ equations already stated, the generalisation to include some number, Γ, of D-branes is straightforward. One must simply keep non-zero the integration constant at the last step of the previous subsection! So the general string equation, which includes world sheet boundaries in the $z \to +\infty$ perturbative expansion, is:

$$\int dz\, \mathcal{R}^i\mathcal{D}_2^{ij}\mathcal{R}^j = \nu^2\Gamma^2\,. \tag{2.12}$$

The reader might find the above open string formulation a little brief, and thus find it hard to believe that it can be so simple to include D-brane sectors. There is in fact an elegant underlying structure to all of this which is being suppressed. First, from a direct matrix model computation, for the $(2, 2k-1)$ case the perturbative open string physics can be explicitly recovered by including a term $\frac{\Gamma}{N}\log(1-M^2)$ in the one-matrix model of

[8] In fact, the string equations of ref. [21]–[24], for which the general (q,p) case was formulated in terms of integrable hierarchies in ref. [45], can be written as the first integrals of the $q-1$ equations $\mathcal{D}_1^{ij}\mathcal{R}^j = 0$. The $t_{i-1,0}$ arise as integration constants, and so in our conventions those string equations are just $\mathcal{R}^j = 0$. The non-perturbative problems of these equations are well-known. In the present formulation, these equations are taken as only perturbatively true, the correct non-perturbative physics being supplied by equation (2.11).

JHEP03(2004)041

the $N \times N$ matrix M, as was carried out originally in ref. [25, 26]. The result is in fact:

$$\mathcal{R} = \frac{1}{2}\nu\Gamma\hat{R}\,, \tag{2.13}$$

where $\hat{R}$ is the resolvent of the differential operator $Q = d^2 - u$, defined as a formal inverse [46, 44]:

$$\hat{R}(z,\sigma)\cdot(Q-\sigma) = 1\,, \qquad i.e., \qquad \hat{R}(z,\sigma) = \langle z|\frac{1}{\nu^2\partial_z^2 - u - \sigma}|z\rangle\,. \tag{2.14}$$

In fact, the differential polynomials are simply coefficients in the expansion in the spectral parameter σ:

$$\hat{R}(z,\sigma) = \sum_{k=0}^{\infty}\frac{R_k[u]}{\sigma^{k+\frac{1}{2}}}\,, \tag{2.15}$$

and the resolvent itself satisfies a differential equation [46]:

$$(u+\sigma)\hat{R}^2 + \frac{1}{2}\hat{R}\hat{R}'' - \frac{1}{4}\left(\hat{R}'\right)^2 = 4\,, \tag{2.16}$$

which is in fact the first integral of the equation $\hat{R}(\mathcal{D}_2 + \sigma\mathcal{D}_1)\hat{R} = 0$, and the integration constant is set by the normalisation of the R_k. Henceforth, we will set the spectral parameter σ to zero.[9] So one can recover the full perturbation theory of solutions of equation (2.13) by simply expanding the resolvent to the desired order in equation (2.16). The non-perturbative physics comes from the full equation, obtained simply by solving equation (2.13) for the resolvent and substituting into equation (2.16). The result is of course our equation (1.1), and can in fact be derived directly in an appropriately tuned complex matrix model or two-cut hermitean matrix model.

The general structure was recognised in ref. [40] and exploited to derive the full open string generalisation above. Briefly explained, the relevant resolvent defines a family of "jet coefficients", $\hat{R}^i$ which are like $q-1$ generalisations of the single resolvent $\hat{R}$ from before [44]. There is a resolvent equation which is simply the first integral of $\hat{R}^i\left(\mathcal{D}_2^{ij} + \sigma\mathcal{D}_1^{ij}\right)\mathcal{R}^j = 0$, where the integration constant is in fact q^2 in the normalisation we chose earlier. The open string generalisation of perturbation theory is simply $\mathcal{R}^j = \nu\Gamma\hat{R}^j/q$, and the full non-perturbative physics follows again by using the larger equation after substituting for the resolvent, which is what we wrote in equation (2.12). Much more about the structure of the equations can be found in ref. [40]. Note also that an explicit two-matrix model realisation of an example of this more general formulation was derived explicitly in ref. [41].

2.3 The proposal

The proposal is simple. The string equations presented above are to be taken as a definition of the type 0A string theory in less than two dimensions, in a particular family of backgrounds. In particular the backgrounds are the (q,p) superconformal minimal models, and

[9]In fact, σ has considerable significance in this context, as it couples to a boundary operator. Its effects are interesting [47], but we shall not explore them here.

JHEP03(2004)041

the "non-critical" stringy physics is realised by coupling them to super-Liouville theory. The Liouville coupling is done through the lowest dimension operator in the theory, with coupling given by z. In the large positive z perturbative world-sheet expansion, there will be boundaries, and Γ represents the number of background D-branes in the model. It is expected that large negative z will give a closed string expansion with insertions of Γ units of R-R flux. The string equations have smooth (and probably unique in general) solutions which connect these asymptotic behaviours to each other through the non-perturbative region of small z.

The bulk operators in the theory are readily identified in the q-KdV formalism. One picks a particular theory by picking L and K such that $Kq+L=p$ and for this choice, set

$$\left(K+\frac{L}{q}\right)t_{l,k}=\delta_l^L\delta_k^K\,. \tag{2.17}$$

All higher $t_{l,k}s$ are set to zero. All the lower ones will be couplings to operators.[10] In particular, the family of $q-1$ operators $t_{l,0}$ are very natural in the theory, since $z=-t_{1,0}$ is the coupling to the lowest dimension operator, and its cousins $\mathcal{B}_i=-t_{i,0}$ have an interpretation as a family of generalised fields such as the magnetic field (see below). We shall study some examples below.

The evidence for the case $q=2, p=2m$ was already presented in ref. [30]. This proposal extends that work further to all of the superconformal minimal models in a natural way, and it will certainly be interesting to study the role (in this new context) of the rich structures which determine the theory's internal organisation, such as the generalised q-KdV flows, the $W^{(q)}$-algebra of constraints, the language of twisted bosons, *etc.*

3. Dressed superconformal minimal models and their bosonic cousins

In proposing that the non-perturbative string physics found in the string equation (1.1) actually pertains to the type 0A string theory, the operators of the bosonic string theory (the $(2,2k-1)$ conformal minimal models coupled to gravity) had to be mapped onto the operator content of the $(2,4k)$ superconformal minimal models, dressed with gravity. This was carried out in ref. [30]. This is in itself an interesting fact, and if the proposal of this paper is to be believed, we must show that this is the case more generally since there is the danger that the correspondence stops with that sub-series.

First, a brief reminder of the bosonic case is in order. (A review is given in ref. [49], for example.) Consider the (q,p) theory. It has central charge given by

$$c=1-6\frac{(p-q)^2}{pq}\,. \tag{3.1}$$

The bare operators are labelled by two integers, r and s, satisfying $1\leq r\leq q-1$, $1\leq s\leq p-1$, with $pr\geq qs$, and shall be denoted $\mathcal{O}^0_{r,s}$. Their dimensions in ordinary conformal field theory ("bare dimensions") are:

$$h_{r,s}=\frac{(pr-qs)^2-(p-q)^2}{4pq}\,. \tag{3.2}$$

[10] One of them is in fact a boundary operator and not a bulk field at all [48].

After coupling to gravity, the operator is dressed by the Liouville field φ according to

$$\mathcal{O}_{r,s} = e^{\beta_{r,s}\varphi}\mathcal{O}^0_{r,s}\,, \tag{3.3}$$

and conformal invariance determines the dressing to be:

$$\beta_{r,s} = \frac{p+q-(pr-qs)}{\sqrt{2pq}}\,. \tag{3.4}$$

It turns out that the physics of the operators which arise from the doubled scaled approach is as follows: In unitary models ($p = q+1$), the Liouville coupling is through the cosmological constant, which couples to the area while more generally, it is to the smallest dimension operator in the theory. From the above, this is when $pr - qs = 1$, and so the Liouville dressing of this operator is $\beta = (p+q-1)/\sqrt{2pq}$. The dimension which we really care about for the operator $\mathcal{O}_{r,s}$ is its Liouville scaling relative to the smallest dimension operator. This is:

$$\Delta_{r,s} = \frac{\beta_{r,s}}{\beta} = \frac{p+q-(pr-qs)}{p+q-1}\,. \tag{3.5}$$

Turning to the superconformal (q,p) series, the operators are labelled in the same way as before. However, there are two sectors: If $r - s$ is even the operator is from the NS sector while it is from the R sector otherwise. The central charge is:

$$\hat{c} = 1 - 2\frac{(p-q)^2}{pq}\,, \tag{3.6}$$

and the bare weights of operators are:

$$h_{r,s} = \frac{(pr-qs)^2-(p-q)^2}{8pq} + \frac{1-(-1)^{p-q}}{32}\,. \tag{3.7}$$

The Liouville weights are

$$\beta_{r,s} = \frac{p+q-(pr-qs)}{2\sqrt{pq}}\,. \tag{3.8}$$

Note that the R sector operator with vanishing $pr - qs$ is in fact the R ground state, with the familiar weight $h = \hat{c}/16$. In identifying the lowest dimension operator for the next step, it is important to distinguish between the two sectors, and to note the restrictions on the allowed (q,p). The point is that $p > q$, and p and q are either both odd or both even. In the odd case, they must be coprime, while in the even case $p/2$ and $q/2$ must be coprime. So the lowest dimension operator in the case when they are even comes from the NS sector, with $\beta = (p+q-2)/\sqrt{2pq}$. This was the case identified in ref. [30], whence they further specialised to the case ($q = 2, p = 4k$), and recovered the same spectrum as the $(2, 2k-1)$ bosonic minimal models. The general formula for even (q,p) for the relative Liouville weight of operator $\mathcal{O}_{r,s}$ is:

$$\Delta_{r,s} = \frac{p+q-(pr-qs)}{p+q-2}\,, \tag{3.9}$$

JHEP03(2004)041

from which, after cancelling various factors of two, seeing that $r=1$ only, gives the familiar result for the scaling of the operator $\mathcal{O}_j$ coupling to t_j in the kth model:

$$\Delta_j = 1 - \frac{j}{k}\,. \tag{3.10}$$

The formula of equation (3.9) stands for all the even cases, and it should be clear that there will always be factors of two cancelling on the top and bottom lines to allow an embedding of the bosonic operator spectrum (3.5) into the supersymmetric one. There remains the case of odd (q,p). Here the lowest dimension operator is in fact from the R sector, and we see that $\beta=(p+q-1)/\sqrt{2pq}$. Therefore the relative Liouville weight of operator $\mathcal{O}_{r,s}$ is:

$$\Delta_{r,s} = \frac{p+q-(pr-qs)}{p+q-1}\,, \tag{3.11}$$

There are no cancellations of factors of 2 required here, and we see that this has exactly the same form as the generic bosonic formula (3.5). This is an interesting and fortuitous fact which gives weight to the main proposal of this paper.

In what follows, we shall focus on a particular set of examples to check our understanding, and the precise content of the proposal.

4. The 3-series

We are now ready to unpack a particular example and play with it. Let us focus on the case $(3,p)$. The objects we need are the differential polynomials from the 3-KdV (or Bousinnesq) hierarchy, or equivalently, the structures $\mathcal{D}^{ij}_{1,2}$ which can be used to generate them by recursion starting from $R^2_{1,0}=3=R^3_{2,0}$, $R^2_{2,0}=0=R^3_{1,0}$.

Choosing

$$Q = d^3 + \frac{3}{4}\{u_2,d\} + u_3\,, \tag{4.1}$$

we have:

$$\begin{aligned}
\mathcal{D}^{22}_2 &= \frac{2}{3}d^3 + \frac{1}{2}u_2' + u_2 d\,, \qquad \mathcal{D}^{23}_2 = u_3 d + \frac{2}{3}u_3'\,, \qquad \mathcal{D}^{32}_2 = u_3 d + \frac{1}{3}u_3'\,,\\
\mathcal{D}^{33}_2 &= -\frac{1}{18}d^5 - \frac{5}{12}u_2 d^3 - \frac{5}{8}u_2' d^2 + \left(-\frac{1}{2}u_2^2 - \frac{3}{8}u_2''\right)d + \left(-\frac{1}{2}u_2u_2' - \frac{1}{12}u_2'''\right),
\end{aligned} \tag{4.2}$$

and, more simply

$$\mathcal{D}^{22}_1 = 0 = \mathcal{D}^{33}_1\,, \qquad \mathcal{D}^{23}_1 = d = \mathcal{D}^{32}_1\,. \tag{4.3}$$

We will need the explicit expressions for the first three rungs $k=0,1,2$ on the two ladders making up the hierarchy of $R^i_{l,k}$ in order to construct our examples. They can be determined to be (after two applications of the $\mathcal{D}_2$ operator according to the right hand side of equation (2.6), integrating the resulting total derivatives each time):

$$\begin{aligned}
R^2_{1,0} &= 3\,, \qquad R^3_{1,0} = 0\,,\\
R^2_{2,0} &= 0\,, \qquad R^3_{2,0} = 3\,;\\
R^2_{1,1} &= u_3\,, \qquad R^3_{1,1} = \frac{3}{2}u_2\,,
\end{aligned}$$

$$
\begin{aligned}
R^2_{2,1} &= -\frac{1}{4}\left(u_2'' + 3u_2^2\right)\,, \qquad R^3_{2,1} = 2u_3\,;\\
R^2_{1,2} &= -\frac{1}{12}u_2'''' - \frac{3}{4}u_2u_2'' - \frac{3}{8}(u_2')^2 - \frac{1}{2}u_2^3 + \frac{2}{3}u_3^2\,, \quad R^3_{1,2} = \frac{2}{3}\left(u_3'' + 3u_3u_2\right)\,,\\
R^2_{2,2} &= -\frac{1}{9}u_3'''' - \frac{5}{6}u_2u_3'' - \frac{5}{12}u_2''u_3 - \frac{5}{12}u_2'u_3' - \frac{5}{4}u_2^2u_3\,,\\
R^3_{2,2} &= -\frac{1}{6}u_2'''' - \frac{5}{4}u_2u_2'' - \frac{15}{16}(u_2')^2 - \frac{5}{8}u_2^3 + \frac{5}{3}u_3^2\,.
\end{aligned}
\tag{4.4}
$$

The string equation is rather long when written out, and is [40]:

$$
\begin{aligned}
&\frac{1}{2}u_2\mathcal{R}_2^2 + \frac{2}{3}\mathcal{R}_2\mathcal{R}_2'' - \frac{1}{3}\left(\mathcal{R}_2'\right)^2 + u_3\mathcal{R}_2\mathcal{R}_3 - \frac{1}{18}\left(\mathcal{R}_3\mathcal{R}_3'''' - \mathcal{R}_3'\mathcal{R}_3''' + \frac{1}{2}\left(\mathcal{R}_3''\right)^2\right) -\\
&\qquad -\frac{5}{12}\left(u_2\mathcal{R}_3\mathcal{R}_3'' - \frac{1}{2}u_2\left(\mathcal{R}_3'\right)^2 + \frac{1}{2}u_2'\mathcal{R}_3\mathcal{R}_3'\right) - \frac{1}{12}\left(3u_2^2 + u_2''\right)\mathcal{R}_3^2 = \nu^2\Gamma^2
\end{aligned}
\tag{4.5}
$$

The indices on the $\mathcal{R}^i$ have been temporarily dropped from upper to lower to permit a tidier equation to be written. The content of the $\mathcal{R}^i$ will be determined by which model is chosen.

4.1 The (3,4) (bosonically dressed ising) model as a subsector

Actually, the model that is of interest to us is the $(3,5)$ model, but first we shall look at the $(3,4)$ model. In the bosonic system, this is in fact the unitary model of the $(3,*)$ series, and is the critical Ising model. Its string equations (in the other formalism) were derived and studied in refs. [50]–[52]. The non-perturbative physics of the closed string equations formulated as reviewed in section 2 was studied in refs. [39, 42].

In this new context, this model does not exist in its own right, and will be a subsector of the unitary model of interest, the (3,5), which is in fact the tricritical Ising model coupled to supergravity. We will be able to switch on an operator in the (3,5) to approach this (3,4) model by RG flow. We start with the (3,4) since it is a familiar example, and it will help us check a crucial normalisation we will need later. The basic model has fixed $t_{1,2} = 3/7$ and all other higher $t_{l,k} = 0$, and:

$$
\begin{aligned}
\mathcal{R}^2 &= R^2_{1,2} + \frac{5}{3}t_{2,1}R^2_{2,1} + \frac{4}{3}t_{1,1}R^2_{1,1} + t_{1,0}\\
\mathcal{R}^3 &= R^3_{1,2} + \frac{5}{3}t_{2,1}R^3_{2,1} + \frac{4}{3}t_{1,1}R^3_{1,1} + 2t_{2,0}\,.
\end{aligned}
\tag{4.6}
$$

If this were a viable model in its own right (it is not in this context), we can check the scalings of the operator content by working directly at the level of the sphere (tree level). One can drop all derivatives in this limit, and recall that in perturbation theory that $\mathcal{R}^2 = 0$ and $\mathcal{R}^3 = 0$. Since at this level we have:

$$
R^2_{1,2} \sim u_2^3 \sim u_3^2\,, R^2_{2,1} \sim u_2^2\,, R^2_{1,1} \sim u_3\,, R^3_{1,2} \sim u_2u_3\,, R^3_{2,1} \sim u_3\,, R^3_{1,1} \sim u_2\,,
\tag{4.7}
$$

we can determine the z scaling of all of the operators:

$$
-t_{1,0} = z \sim z^1\,, \qquad \mathcal{B} = -2t_{2,0} \sim z^{5/6}\,, \qquad t_{2,1} \sim z^{1/3}\,.
\tag{4.8}
$$

Note that $t_{1,1} \sim z^{1/2}$ and can be absorbed into a boundary operator. The numbers in equation (4.8) are indeed the dressed dimensions relative to the cosmological constant $z \sim \mu$ in the model. Recall that this is all in the context of the *bosonic* superconformal series, and one can confirm the numbers by comparing to the continuum result given in equation (3.5).

In the present context these numbers are irrelevant. The operator dimensions will be recomputed in the next subsection once the model is subsumed into the (3,5) model, which in the context of the superconformal series, is indeed the unitary model.

Next, we check a normalisation. The basic equations in perturbation theory, for the case $\Gamma = 0$, can be chosen to be $\mathcal{R}^2 = 0 = \mathcal{R}^3$, where:

$$\mathcal{R}^2 = -\frac{1}{12}u_2'''' - \frac{3}{4}u_2 u_2'' - \frac{3}{8}(u_2')^2 - \frac{1}{2}u_2^3 + \frac{2}{3}u_3^2 - z\,; \qquad \mathcal{R}^3 = \quad R^3_{1,2} = \frac{2}{3}\left(u_3'' + 3u_3u_2\right) - \mathcal{B}\,. \tag{4.9}$$

On the sphere, we have:

$$-\frac{1}{2}u_2^3 + \frac{2}{3}u_3^2 = -z\,, \qquad 2u_3u_2 = \mathcal{B}\,, \tag{4.10}$$

and we may solve the second equation, and substitute into the first, to get

$$u_3 = \frac{\mathcal{B}}{2u_2}\,, \qquad 3u_2^5 - \mathcal{B}^2 = 6zu_2^2\,. \tag{4.11}$$

The second polynomial equation can solved for $u_2(z,\mathcal{B})$, and then using this the first equation then yields $u_3(z,\mathcal{B})$. A simple solution arises when $\mathcal{B} = 0$, for which it is seen that $u_3 = 0$. This is appropriate, since $\mathcal{B}$ is actually the magnetic field, and u_3 is the response to it, i.e., the magnetisation.

Staying in the sector where $u_3 = 0$ and $\mathcal{B} = 0$, and keeping $\Gamma = 0$, we get (writing $u = -u_2$):

$$\mathcal{R}^2 = \frac{1}{12}u'''' - \frac{3}{4}uu'' - \frac{3}{6}(u')^2 + \frac{1}{2}u^3 - z\,; \qquad \mathcal{R}^3 = 0\,, \tag{4.12}$$

(where the last equation is an identity), and the vanishing of $\mathcal{R}^2$ will yield perturbation theory for the function $u(z)$. After a few lines of algebra, the first few terms of the solution are:

$$u(z) = -2^{1/3}z^{1/3} - \frac{\nu^2}{12z^2} + \cdots\,, \tag{4.13}$$

and so if we use equation (1.3), we reproduce the standard result that the torus partition function for the gravitating Ising model is:

$$Z = -\frac{1}{24}\frac{(p-1)(q-1)}{(p+q-1)}\log z \to -\frac{1}{24}\log z\,. \tag{4.14}$$

This sets our normalisation for the model of interest in the next subsection.

Let us now work with $\Gamma \neq 0$ to see what we might learn. The expansion above in equation (4.13) can be taken for either large positive z or negative z. Interestingly, switching on Γ breaks this symmetry, in that the large positive z expansion picks up terms linear in Γ which are imaginary. So if we wish for $u(z)$ to be real, and to smoothly connect

JHEP03(2004)041

to non-zero Γ (which seems reasonable) then we are forced to pick the expansion as being for the negative z direction. The fact that the other choice does not smoothly connect onto non-zero Γ, while keeping $u(z)$ real, may well be an earmark of a branch of new physics,[11] but we will not explore that here.

So, for $\Gamma \neq 0$, expanding in large negative z, we find

$$u(z) = 2^{1/3} z^{1/3} - \frac{2^{2/3}\nu\Gamma}{z^{5/6}} + \frac{1}{12}\frac{\nu^2}{z^2}\left(4\Gamma^2 - 1\right) - \frac{91}{324}\frac{2^{1/3}\nu^3}{z^{19/6}}\Gamma\left(\Gamma^2 - 1\right) + \cdots , \qquad (4.15)$$

while expanding for large positive z we find

$$\begin{aligned} u(z) \;=\; & -\frac{2}{3}\frac{\nu^2}{z^2}\left(3\Gamma^2+1\right) + \frac{16}{81}\frac{\nu^8}{z^9}\left(3\,\Gamma^2+16\right)\left(3\,\Gamma^2+49\right)\left(3\,\Gamma^2+4\right)\left(3\,\Gamma^2+1\right)\times \\ & \times\left\{1 - \frac{10}{9}\frac{\nu^6}{z^7}\left(27\,\Gamma^6 + 2079\,\Gamma^4 + 44982\,\Gamma^2 + 246424\right)\right\} + \cdots . \end{aligned} \qquad (4.16)$$

Based on our experience with these equations, it is entirely reasonable to expect the string equation supplies the means to uniquely connect these two expansions smoothly through the non-perturbative region. It would be interesting to prove this analytically.

Integrating up twice to get the free energy, we see that the natural dimensionless world-sheet expansion parameter (i.e., the string coupling) is $\kappa = \nu/z^{7/6}$. The topological expansion is very interesting. For large negative z, we see that the first two terms in equation (4.15) represent the sphere and the disc respectively, while the next term contains the torus and the cylinder, and the next the holey torus, and so forth. The interpretation is that there are Γ D-branes (in some units), and there is an appropriate factor of Γ^b for every boundary, accompanying the usual κ^{2h-2+b} in string perturbation theory.

For large positive z, things are rather different. With the same identification of expansion parameters as before, the presence of only even powers of Γ presents a puzzle. The first non-vanishing physics is at the level of the torus and the cylinder, but then nothing until higher orders. These orders are the four-torus (sphere with four handles), holey sphere (it has six holes), *etc.* There are no terms like the disc, *etc.,* with odd numbers of boundaries. A simpler explanation, following ref. [30], is to not attribute a brane interpretation to this regime at all. Rather, there are all the correct surfaces present for purely *closed string* perturbation theory, and instead, each power of Γ^2 represents an insertion of R-R flux. It would be interesting to examine an underlying matrix model (such as the one in ref. [41]) further to add evidence for this claim. In the present case, we shall take this as the simplest interpretation. We could invoke other more complicated reasons for the vanishing of the surfaces with odd numbers of boundaries, but none have occurred to us which seem as natural as this one.

With this interpretation, it is tempting to declare that we have found our first new example of a transition between closed string physics with fluxes and physics with D-branes present. Bear in mind however that this (3,4) model is not a complete 0A model in its own right, since it is not a member of the superconformal series (since 3 is odd but 4 is not).

[11] We could also continue Γ to $i\Gamma$, which is interesting, but we shall keep a definite choice of Γ in order to compare each member of the 3-series with each other.

However, it will be (with appropriate coupling to an operator) naturally embedded within the prototype complete 0A model we study in the next subsection, and so the remarks here about R-R flux, and the possibility of a non-perturbative geometric transition from flux to D-branes, will become an honest interpretation within that context.

4.2 The (3,5) model: superdressed tricritical ising

It is quite amusing that the first complete non-trivial model of interest to us is the tricritical Ising model. This model is fascinating in its own right since in ordinary conformal field theory (i.e., no dressing) it is the unique minimal model which is both conformal and superconformal [53]. In the bosonic context, it appears gravitationally dressed as the $(4,5)$ model. In this context, it is supergravitationally dressed and is the $(3,5)$.

The model has fixed $t_{2,2}=3/8$ and all other higher $t_{l,k}=0$, and:

$$\begin{aligned}
\mathcal{R}^2 &= R^2_{2,2}+\frac{7}{3}t_{1,2}R^2_{1,2}+\frac{5}{3}t_{2,1}R^2_{2,1}+\frac{4}{3}t_{1,1}R^2_{1,1}+t_{1,0}\\
\mathcal{R}^3 &= R^3_{2,2}+\frac{7}{3}t_{1,2}R^3_{1,2}+\frac{5}{3}t_{2,1}R^3_{2,1}+\frac{4}{3}t_{1,1}R^3_{1,1}+2t_{2,0}\,.
\end{aligned}\tag{4.17}$$

To compute the operator dimensions, we note that in addition to the sphere level behaviour listed in equation (4.7), we have:

$$R^2_{2,2}\sim u_2^2u_3 \qquad R^3_{2,2}\sim u_2^3\sim u_3^2\,.\tag{4.18}$$

From the second equation we see that $u_2\sim\mathcal{B}^{1/3}$ and $u_3\sim\mathcal{B}^{1/2}$, and so a little algebra gives the z scaling of all of the operators:

$$-t_{1,0}=z\sim z^1\,,\qquad \mathcal{B}=-2t_{2,0}\sim z^{6/7}\,,\qquad t_{1,2}\sim z^{1/7}\,,\qquad t_{1,1}\sim z^{4/7}\,.\tag{4.19}$$

Note that $t_{2,1}\sim z^{3/7}$ and can be absorbed into a boundary operator. The operator scalings in equation (4.19) are precisely the scalings required by the formula (3.11) (derived with continuum methods) to work, and the model is to be interpreted as a type 0A background, as per our proposal.

Again, let us study perturbation theory with $\Gamma=0$ first. Then we have a choice $\mathcal{R}^3=0=\mathcal{R}^2$ where now (switching off $t_{1,1}$):

$$\begin{aligned}
\mathcal{R}^2 &= -\frac{1}{9}u_3''''-\frac{5}{6}u_2u_3''-\frac{5}{12}u_2''u_3-\frac{5}{12}u_2'u_3'-\frac{5}{4}u_2^2u_3-z\,,\\
\mathcal{R}^3 &= -\frac{1}{6}u_2''''-\frac{5}{4}u_2u_2''-\frac{15}{16}(u_2')^2-\frac{5}{8}u_2^3+\frac{5}{3}u_3^2-\mathcal{B}\,,
\end{aligned}\tag{4.20}$$

and so on the sphere we have the equations

$$-\frac{5}{4}u_2^2u_3=z\,,\qquad \frac{5}{3}u_3^2=\mathcal{B}\,.\tag{4.21}$$

We may solve the first equation and substitute into the second to get:

$$u_3=-\frac{4}{5}\frac{z}{u_2^2}\,,\qquad \frac{5}{8}u_2^7+\mathcal{B}u_2^4=\frac{16}{15}z^2\,.\tag{4.22}$$

Again, this system can be used to solve for $u_2(z,\mathcal{B})$ and $u_3(z,\mathcal{B})$. Notice that this time (in contrast to the case in the previous section) we can choose $\mathcal{B}=0$ and still retain non-zero u_3. In fact, after a bit of algebra, we find for this choice, writing ($u_2=-u$):

$$u=-\frac{2}{75}(75)^{6/7}z^{2/7}\,,\qquad u_3=-\frac{1}{5}(75)^{2/7}z^{6/7}\,. \tag{4.23}$$

Let us now go beyond tree level to uncover a bit more physics, staying with $\Gamma=0$ for now. More algebra yields from our equations the next few levels of closed string perturbation theory:

$$\begin{aligned} u &= -\frac{2}{75}(75)^{6/7}z^{2/7}-\frac{2}{1029}\frac{\nu^2}{z^2}+\frac{240242}{5294205}(75)^{1/7}\frac{\nu^4}{z^{30/7}}+\cdots\,,\\ u_3 &= -\frac{1}{5}(75)^{2/7}z^{6/7}+\frac{24}{1715}75^{3/7}\frac{\nu^2}{z^{13/7}}+\frac{116003}{8823675}(75)^{4/7}\frac{\nu^4}{z^{29/7}}+\cdots\,. \end{aligned} \tag{4.24}$$

We can readily integrate the above result for u twice to get the expression for the free energy, using equation (1.3). We see that the dimensionless world sheet expansion parameter (string coupling) is $\kappa=\nu/z^{8/7}$.

Now we switch Γ back on, and solve the full string equation (4.5) perturbatively, starting with the same sphere level solutions. We find that the next non-vanishing level of perturbation theory is of order ν, and denoting the contributions to u and u_3 at this order by f and g, respectively, we have the following equation for them:

$$f(z)=\frac{(75)^{6/7}}{7875z^{14}}\left\{-7(75)^{5/7}z^{97/7}g(z)\pm5\cdot7^{1/7}\sqrt{(75)^{1/7}z^{184/7}\left(49(75)^{2/7}z^{10/7}g(z)^2+36\Gamma^2\right)}\right\}. \tag{4.25}$$

This dreadful equation can be simplified by realising that the solution for $g(z)$ must reduce to zero at $\Gamma=0$, and similarly for $f(z)$, since we would like to match onto closed string perturbation theory in this limit. One way that we can achieve this is by setting the square root to zero, since $g(z)\propto\Gamma$ for this choice, and we remark in passing that other choices of $g(z)$ will given two other branches of solutions. It would be interesting to explore these further. In proceeding to develop the non-zero Γ perturbation theory, we note that (rather like happened in the case of the previous subsection) in order to have real solutions for u and u_3, we must take our expansion as pertaining to negative z. Carrying on with the choice in equation (4.25) that makes the square root vanish, after going on to the next few levels of perturbation theory we have:

$$\begin{aligned} u &= -\frac{2}{75}(75)^{6/7}z^{2/7}\mp\frac{2}{35}(75)^{3/7}\frac{\nu\Gamma}{z^{4/7}}-\frac{2}{1029}\frac{\nu^2}{z^2}+0\ \nu^3+\frac{240242}{5294205}(75)^{1/7}\frac{\nu^4}{z^{30/7}}+\cdots\,,\\ u_3 &= -\frac{1}{5}(75)^{2/7}z^{\frac{6}{7}}\pm\frac{6}{7}(75)^{-1/7}\frac{\nu\Gamma}{z^{5/7}}+\frac{3}{1715}(75)^{3/7}\frac{\nu^2(21\Gamma^2+8)}{z^{13/7}}+\frac{52}{2401}\frac{\nu^3\Gamma}{z^3}-\\ &\quad-\frac{(75)^{4/7}}{8823675}\frac{\nu^4}{z^{29/7}}\left(33075\Gamma^4-300720\Gamma^2-116003\right)+\cdots\,. \end{aligned} \tag{4.26}$$

It is very curious that there is no contribution to u of surfaces with boundary beyond the disc, although this is not the case for u_3. Similarly interesting is the choice to make $g(z)$ vanish in equation (4.25). In that case, it is the u_3 perturbative series which fails to be afflicted by Γ at higher orders, while the u perturbative series has contributions from all orders. However, it is clear that for generic choices of $g(z)$, we will have contributions to the physics at all orders in perturbation theory from all types of surfaces with and without boundaries.

The reader should not be concerned that these choices for $g(z)$ correspond to non-perturbative ambiguities in the physics. There is nothing about the properties of the string equation which suggest that this is the case, as far as we know. It must be recalled that there are also constraints that cannot be specified in the perturbative analysis, such as the nature of the asymptotic expansion in the other large z regime. Generically, we expect that the requirement of fixing to perturbation theory in *both* the $z \to +\infty$ and the $z \to -\infty$ regimes will fix things uniquely. The present analysis is simply a demonstration of the perturbative possibilities available to us.

Turning to a wider range of choices, note that we can start at sphere level with vanishing $u = -u_2$. This is now physics that the equations $\mathcal{R}^3 = 0 = \mathcal{R}^2$ cannot handle, but the full string equation can, since the sphere level condition is:

$$\frac{1}{2}u_2(\mathcal{R}^2)^2 + u_3\mathcal{R}^2\mathcal{R}^3 - \frac{1}{4}u_2^2(\mathcal{R}^3)^2 = 0\,, \tag{4.27}$$

where

$$\mathcal{R}^2 = -\frac{5}{4}u_2^2u_3 - z \qquad \text{and} \qquad \mathcal{R}^3 = \frac{5}{3}u_3^2\mathcal{B}\,. \tag{4.28}$$

The choice $u_2 = 0$ leaves:

$$u_3 = \pm\frac{(15)^{1/2}}{5}\mathcal{B}^{1/2}\,. \tag{4.29}$$

We can choose $\mathcal{B}$ to vanish, as a special case, resulting in vanishing u_3 on the sphere. There are no doubt solutions for which u_3 is non-zero beyond tree level, but we won't pursue those here.

It is also interesting to consider the case of u_3 vanishing to all orders and beyond. In such a case, we have in the string equation the functions:

$$\mathcal{R}^3 = \frac{1}{6}u'''' - \frac{5}{4}uu'' - \frac{15}{16}(u')^2 + \frac{5}{8}u^3\,; \qquad \mathcal{R}^2 = -z\,. \tag{4.30}$$

Expanding, we have for example in the $z \to +\infty$ limit:

$$\begin{aligned} u(z) = & -\frac{2}{3}\frac{\nu^2\left(3\,\Gamma^2+1\right)}{z^2} - \frac{50}{6561}\frac{\nu^{16}}{z^{18}}\left(3\,\Gamma^2+1\right)^2 \times \\ & \times \left(3\,\Gamma^2+4\right)^2\left(3\,\Gamma^2+25\right)^2\left(3\Gamma^2+64\right)\left(3\Gamma^2+16\right) \times \\ & \times \Bigg\{1 + \frac{100}{2187}\frac{\nu^{14}}{z^{16}}\left(3\,\Gamma^2+1\right)\left(3\,\Gamma^2+4\right)\left(3\,\Gamma^2+25\right) \times \\ & \quad \times \left(162\Gamma^8 + 41931\Gamma^6 + 4316193\Gamma^4 + 187005426\Gamma^2 + 2729467376\right)\Bigg\} + \cdots\,. \end{aligned} \tag{4.31}$$

The physics of this regime is again expected to be connected by smooth non-perturbative physics through the strong coupling region to other perturbative physics, such as that given in equation (4.24), and it would be interesting to pursue this further.

Notice that we can also have a solution for which $u \sim z^{2/7}$ and $u_3 = 0$ at tree level, in the large negative z regime, which is again something which is allowed by the sphere condition (4.27) of the full string equation, and not the restriction to the solutions of $\mathcal{R}^3 = 0 = \mathcal{R}^2$. The first few terms of the asymptotic expansion of the solution of our string equation are:

$$\begin{aligned} u(z) = & -\frac{2}{5}5^{5/7}z^{2/7} + \frac{2\nu^2}{1029}\frac{\left(147\,\Gamma^2 - 43\right)}{z^2} + \frac{4\cdot 5^{2/7}}{1058841}\frac{\nu^4}{z^{30/7}}\left(43218\Gamma^4 - 234318\Gamma^2 + 15389\right) \\ & + \frac{8\cdot 5^{4/7}}{363182463}\frac{\nu^6}{z^{46/7}}\left(6353046\Gamma^6 - 104047335\Gamma^4 + 326118177\Gamma^2 + 5183425\right) + \cdots . \end{aligned} \tag{4.32}$$

Examining this expression and the previous one, tentatively associating each power of Γ with a boundary, we have the sphere, torus and cylinder from the first two terms, but no disc. The next term contains the double torus, torus with two holes and sphere with four holes. Missing are any surfaces with an odd number of holes. Following on from what we have already stated about the expansions observed in the previous subsections, (and based on examples in ref. [30]), this is highly suggestive of a R-R flux interpretation instead of boundaries. So tentatively, there are no world sheet boundaries in the expansion but fluxes: one insertion for every power of Γ^2, however this needs to be explored more.

It is our conjecture that it will *always* be the case that only even powers of Γ appear in solutions for which the behaviour on the sphere is a solution of equation (4.27) but not $\mathcal{R}^3 = 0 = \mathcal{R}^2$, and further, that it extends to all of the string equations for the (q, p) series. A proof of this would be useful to construct.

So we have a number of possibilities, depending upon the boundary conditions. We remind the reader that the overall situation is necessarily more complicated that for the study of the equation (1.1), for the $(2, 4k)$ series because there are two functions to solve for, u_2 and u_3, in addition to a number of couplings, $\mathcal{B}$ and $t_{1,2}$. It is clear that we can have solutions which have D-branes in the large negative z regime, but only fluxes for positive z. These are new fully non-perturbative examples of geometrical transitions. We may well also have solutions which have fluxes in *both* regime, which is certainly novel. These would arise if we choose sphere behaviour in both asymptotic regions which are allowed by the tree level condition (4.27) of the full string equation, but not $\mathcal{R}^3 = 0 = \mathcal{R}^2$. We do not know if this is consistent non-perturbatively, but certainly seems like a possibility at the level of our current analysis.

For either the former or the latter case, we can always expect transition phenomena for another reason: We must not forget that we have switched off a number of operators in the theory. Recall from the explorations in the last subsection of the (3,4) model, that the large negative z region contained branes, while the large positive z direction contained only fluxes. There was a geometric transition as one went through the strong coupling regime. If we switch on the operator $\mathcal{O}_{1,2}$ by turning on $t_{1,2}$ it will mix the physics we have seen here with the physics of the (3,4) model. So in fact, we have a new way of generating transition phenomena, by turning on $\mathcal{O}_{1,2}$, even if we started with a solution in the (3,5) model with no transition.

Let us see how this must work, continuing to keep $u_3 = 0$ and $\mathcal{B} = 0$. We must now solve the string equation with:

$$\mathcal{R}^3 = \frac{1}{6}u'''' - \frac{5}{4}uu'' - \frac{15}{16}(u')^2 + \frac{5}{8}u^3\,; \qquad \mathcal{R}^2 = \frac{7}{3}t_{1,2}\left(\frac{1}{12}u'''' - \frac{3}{4}uu'' - \frac{3}{8}(u')^2 + \frac{1}{2}u^3\right) - z\,. \tag{4.33}$$

Unfortunately, this becomes a rather difficult problem to proceed with analytically. Already at the sphere, after working to linear order in $t_{1,2}$, we see that the physics is controlled by the solution to this equation:

$$0 = -\frac{25}{256}u^7 - \frac{1}{2}z^2 + \frac{7}{3}zt_{1,2}\left(\frac{1}{2}u^3 - z\right). \tag{4.34}$$

The next step will be to solve this for u at linear order in $t_{1,2}$ and then proceed. It is not clear if the mixing with new world sheets with boundaries will ever be seen in the linear approximation however, or even to any order in perturbation in $t_{1,2}$, so this approach may be doomed. However, we do know that ultimately, since $t_{1,2}$ is a good operator in the theory there will be a flow to the behaviour of the previous subsection givign a new handle with which to switch on transition phenomena.

It is also possible that switching on the other coupling which we have set to zero ($\mathcal{B}$), together with the function $u_3(z)$ for large positive z, may also allow for odd powers of Γ to appear, giving us even more directions in which we can move to find transitions. It is also likely that there is an interesting geometrical description of this moduli space of directions giving a nicer setting in which some of these transition phenomena might have a powerful description. The presence of an underlying supersymmetric model makes this seem worth seeking.

Further study of this $(3,5)$ example ought to be carried out. We've looked at a range of perturbative solutions, and showed that examples of the interesting physics which we seek abound, but further analysis of how these asymptotes connect to each other non-perturbatively ought to be performed by studying the string equation further.

4.3 The torus partition function

Finally, we note two more interesting pieces of data, the one-loop contribution to the partition function. This should give useful universal data which can be compared to the continuum theory. For large $-z$ regime the torus contribution is:

$$Z_A(z) = -\frac{1}{1029}\log z\,, \tag{4.35}$$

while in the $+z$ regime it is:

$$Z_A(z) = -\frac{1}{3}\log z\,. \tag{4.36}$$

The latter result is particularly interesting. First of all, it appears to be a universal contribution yielded by all solutions in the $(3,p)$ series which begin with sphere contributions not satisfying $\mathcal{R}^3 = 0 = \mathcal{R}^2$, but satisfying (4.27). The expected full result [54]:

$$Z_{\rm even} = \frac{1}{2}\left(Z_A(z) + Z_B(z)\right) = -\frac{1}{8}\frac{(p-1)}{(p+1)}\log z\,, \qquad p \text{ odd}\,, \qquad p \neq 0 \mod 3\,, \tag{4.37}$$

JHEP03(2004)041

would seem to require that the 0B contribution in this regime is non-vanishing. (Interestingly, it vanishes exactly for the $(2, 4k)$ series [30], but this had better not happen here since the type 0B contribution is needed to restore the p dependence of the formula. If our proposal is correct, this number serves as a prediction for the type 0B definition (yet to be presented), in order to make contact with the continuum results.

5. Closing remarks

Here, we shall be brief, since we have made the key technical observations along the way. It is hoped that it is apparent to the reader that the old double scaled large N matrix technology, although failing to tell us anything directly about higher dimensional string theories, may still supply useful laboratories within which to study a range of phenomena of current interest. This should be definitely appreciated once it is realised that there are very many well-defined non-perturbative formulations of the physics. These models remain the only real examples we have in string theory where certain questions can be asked and answered at arbitrary values of the coupling. Admittedly, given that the string theories which can be formulated with these methods do not admit a lot of room for a wide variety of physics, the range of questions is rather smaller than for higher dimensional string theory, but there is still scope for lessons to be learned.

Here, we have proposed a natural formulation of the type 0A string in $\hat{c} \leq 1$ backgrounds, the superconformal minimal models coupled to supergravity, extending the work of ref. [30]. This work is all based on non-perturbative formulations of string theory, including both closed and open sectors, which have existed in the literature for over a decade now, but have been recast in a new role as type 0A theories. The first non-trivial new model which we studied in this paper, the (supergravitationally dressed) tricritical Ising model as it turns out, already supplied us with new physics which should be studied further, and presumably there is more to be found in other examples too. There is a lot more to learn about these models, and no doubt the remarkably rich underlying structures (integrable hierarchies, W-algebra constraints and so forth) may play an important role in this new context.

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